Applications of Generalized Universal Valuations

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Paper under the same name is available on arXiv: [2]

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Overall Motivation

- I work with non-commutative, non-totally ordered idempotent semirings which occur naturally in computation [1] and in the mathematical foundations of satellite networking [3].

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- The Giansiaracusas generalized this link to commutative, non-totally ordered idempotent semirings [4]
- We seek to generalize this even further, to establish a geometry of non-commutative idempotent semirings.

Objects of Study

Semirings

Definition

A **semiring** is a set S with a unital (0_S) commutative addition $+_S$ and a potentially non-commutative, unital (1_S) multiplication $*_S$ which distributes over the addition.

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With respect to this partial order we have:

$$\inf(X) = \sum_{x \in X} x$$

Generalized Valuations

From the Giansiracusa's work on tropical schemes[4], we can generalize the notion of a valuation to be over an arbitrary idempotent semiring

Definition

Let R be a ring and Γ an idempotent semiring. We say that a function $\nu : R \to \Gamma$ is a **valuation** if ν is:

Unital: $\nu(0_R) = 0_\Gamma$, $\nu(1_R) = 1_\Gamma = \nu(-1_R)$,

Multiplicative: $\nu(a *_R b) = \nu(a) *_\Gamma \nu(b)$,

Superadditive: $\nu(a +_R b) \geq \nu(a) +_\Gamma \nu(b) = \inf_\Gamma(\nu(a), \nu(b))$.

Universal Valuations

Category of Valuations

Rings with valuation, $(R, \Gamma, R \xrightarrow{\nu} \Gamma)$ form a category, with morphisms being appropriate homomorphisms such that the diagram commutes:

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \nu\downarrow & & \downarrow\nu' \\ \Gamma & \longrightarrow & \Gamma' \end{array}$$

Category of Valuations

If we look at the subcategory where we fix an R:

$$\begin{array}{ccc} R & \xrightarrow{id} & R \\ \nu \downarrow & & \downarrow \nu' \\ \Gamma & \longrightarrow & \Gamma' \end{array}$$

We have a theorem by the Giansiracusas[4]:

Theorem

For a ring R there is a universal valuation $R \to \Gamma_R$ which is initial in the above category.

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The Giansiracusas worked over valuations into commutative semirings but this statement holds even when we allow the semirings to be noncommutative.

Universal Valuations

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In $\mathbb{B}\langle X\rangle$ we have

$$xy \neq yx$$

however in $\mathbb{B}[x]$ we do have

$$xy = yx$$

Universal Valuations

Definition

$$\Gamma_R = \mathbb{B}\langle R\rangle / \sim$$

Where $\sim$ is the congruence generated by the relations

$$x_0 \sim 0 \qquad x_1 \sim 1 \qquad x_{-1} \sim 1$$

$$x_a x_b \sim x_{ab}$$

$$x_{a+b} + x_a + x_b \sim x_a + x_b$$

$$\nu(a) = [x_a]$$

This is almost identical to the construction in [4], except we quotient $\mathbb{B}\langle R\rangle$ rather than $\mathbb{B}[R]$

A Non-Commutative Valuation

What would a non-commutative valuation even look like?

A Non-Commutative Valuation

Let R be the ring of upper triangular 2×2 matrices over $\mathbb{F}_2$. R has eight elements and they are generated by the matrices:

$$i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad j = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad k = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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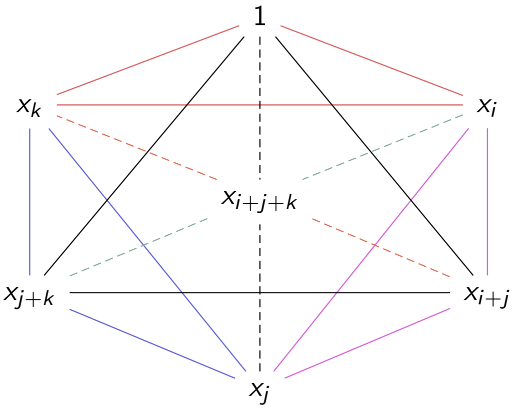
Γ_R consists of $\mathbb{B}$ linear combinations of the elements: $0, 1, x_i, x_j, x_k, x_{i+j}, x_{j+k}, x_{i+j+k}$, with multiplication table given by the multiplication table in R

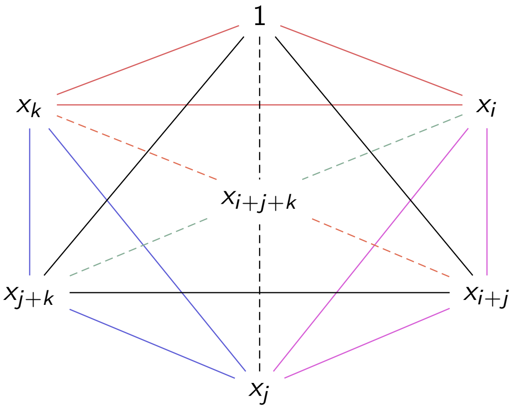
A Non-Commutative Valuation

Γ_R's additive structure can be given diagrammatically:



A Non-Commutative Valuation

This is the Fano plane!



Structure Theorem

We can give an explicit description of the additive structure of Γ_R[2]

Theorem (Structure Theorem for Γ_R)

Let $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ be finite collections of elements in a ring R. In Γ_R we have:

$$\left[\sum_{i\in I} x_{a_i}\right] = \left[\sum_{j\in J} x_{b_j}\right]$$

if and only if $\mathrm{Span}_{\mathbb{Z}}(\{a_i\}_{i\in I}) = \mathrm{Span}_{\mathbb{Z}}(\{b_j\}_{j\in J})$.

Equivalence classes in Γ_R are given by the $\mathbb{Z}$ spans of elements

Applications

Non-Archimedean Case of Ostrowski's Theorem

Corollary

$\Gamma_{\mathbb{Q}} \cong (\mathbb{Z}^{\omega} \cup \{\infty\}, \min, +, \infty, 0)$ *Where* $\nu(a)$ *is the exponents in its prime decomposition if a is nonzero, or* ∞ *otherwise.*

Non-Archimedean Case of Ostrowski's Theorem

Corollary

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Theorem

The Non-Archimedean absolute values on R are in correspondence with:

$$\mathrm{Hom}(\Gamma_R, \mathbb{T})$$

Where $\mathbb{T}$ is the tropical semiring

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Theorem

The Non-Archimedean absolute values on R *are in correspondence with:*

$$\mathrm{Hom}(\Gamma_R, \mathbb{T})$$

Where $\mathbb{T}$ *is the tropical semiring*

Corollary (Non-Archimedean Case of Ostrowski's Theorem)

The Non-Archimedean absolute values on $\mathbb{Q}$ *up to equivalence are the p-adic ones.*

Representations in Ultrametric Spaces

Definition

Let R be a ring and Γ an idempotent semiring. We say a map $\nu : R \to \Gamma$ is a **super-multiplicative valuation** if ν is:

Unital: $\nu(0_R) = 0_\Gamma$, $\nu(1_R) = 1_\Gamma = \nu(-1_R)$

Supermultiplicative: $\nu(a *_R b) \geq \nu(a) *_\Gamma \nu(b)$

Superadditive: $\nu(a +_R b) \geq \nu(a) +_\Gamma \nu(b) = \inf_\Gamma(\nu(a), \nu(b))$

Representations in Ultrametric Spaces

We can form a similar initial super-multiplicative valuation semiring $\widehat{\Gamma}_R$, and we can show that our structure theorem holds:

Theorem

Let $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ be finite collections of elements in a ring R. In $\widehat{\Gamma}_R$ we have:

$$\left[\sum_{i \in I} x_{a_i}\right] = \left[\sum_{j \in J} x_{b_j}\right]$$

If and only if $\mathrm{Span}_{\mathbb{Z}}(a_i) = \mathrm{Span}_{\mathbb{Z}}(b_j)$

Representations in Ultrametric Spaces

Theorem

Let V be an n-dimensional ultrametric space and let $\phi : R \to \mathrm{End}(V)$ be a representation.
ϕ induces a super-multiplicative valuation: $R \to M_n(\mathbb{T})$

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Corollary

If R has a nontrivial n dimensional representation then there is a nontrivial map in:

$$\mathrm{Hom}(\widehat{\Gamma}_R, M_n(\mathbb{T}))$$

Thank you for attending!

In summary:

- We can generalize the notion of a valuation to non-commutative settings.
- We can form an initial object which can be used to classify different structures (non-archimedean absolute values, ultrametric representations, and more)
- We can explicitly describe the additive structure of this object.

Thank you for attending!

Questions?

References

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